In the name of God

Group Theory

(December, 02, 2013)

1. Let G be a group and X be a set. Show that there exists a homomorphism $G \longrightarrow \mathrm{Sym}(X)$ if and only if there exists a function

$$X \times G \longrightarrow X$$
$$(x, g) \mapsto xg$$

such that $x1 = x$ and $x(gh) = (xg)h$, for all $x \in X$ and $g, h \in G$.

2. Show that if G is a finite group of order p^2q^2, where p and q are prime numbers, then G is not simple.

3. Let H be a normal subgroup of a finite group G, such that $(|H|, |G : H|) = 1$. Prove that H has a complement in G.

4. Let G be a finite primitive permutation group on a set X and $1 \neq N \trianglelefteq G$. Then N acts transitively on X. Moreover if N is regular on X, then N is a minimal normal subgroup of G.

5. Suppose that G is a Frobenius group on a set X with kernel K. Show that
(a) $K = \{g \in G \mid \mathrm{Fix}(g) = \emptyset\} \cup \{1\}$, where $\mathrm{Fix}(g) = \{x \in X \mid xg = x\}$.
(b) For all $1 \neq u \in K$, $C_G(u) \subseteq K$; and for all $1 \neq g \in G_x$, $C_G(g) \subseteq G_x$.
(c) $Z(G) = 1$

6. (Ph. D. students) A regular permutation group of finite degree is primitive if and only if it has prime order.

7. (Ph. D. students) Show that every non-abelian group of order 8 is isomorphic to D_8 or Q_8.